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Equality relating Euclidean distance cone to positive semidefinite cone

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Abstract

We know that the cone of Euclidean distance matrices does not intersect the cone of positive semidefinite matrices except at the origin in the subspace of symmetric matrices. Even so, the two cones can be related by an equality.

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1. Background

In the subspace of symmetric matrices $\mathbb{S}$, we know that the convex cone of Euclidean distance matrices $\mathbb{EDM}$ (the EDM cone) does not intersect the positive semidefinite cone $\mathbb{S}_+$ except at the origin, their only vertex; there can be no positive nor negative semidefinite EDM [6].

$$\mathbb{EDM} \cap \mathbb{S}_+ = \mathbf{0} \tag{1}$$

Even so, the two convex cones can be related. We establish an equality

$$\mathbb{EDM} = \mathbb{S}_h \cap (\mathbb{S}_c^{\perp} - \mathbb{S}_+) \tag{2}$$

where

$$\mathbb{S}_h \stackrel{\Delta}{=} \{A \in \mathbb{S} \mid \text{diag}(A) = \mathbf{0}\} \tag{3}$$

is the *symmetric hollow subspace*, and where

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$$\mathbb{S}_c^{\perp} = \{u\mathbf{1}^{\mathrm{T}} + \mathbf{1}u^{\mathrm{T}} | u \in \mathbb{R}^N\} \tag{4}$$

is the orthogonal complement of the *geometric center subspace*

$$\mathbb{S}_c \stackrel{\Delta}{=} \{Y \in \mathbb{S} | Y\mathbf{1} = \mathbf{0}\} \tag{5}$$

In N-dimensional real Euclidean vector space $\mathbb{R}^N$, $\mathbf{1}$ denotes a vector of ones. Equality (2) is not obvious from the various EDM definitions, such as in [5], because inclusion must be proved algebraically. Equality (2) is equally important as the known isomorphisms [2, Section 2] relating the EDM cone to a face of the positive semidefinite cone. But those isomorphisms have never led to this equality relating whole cones $\mathbb{EDM}$ and $\mathbb{S}_+$.

We invoke a matrix variant of the algebraic Schoenberg criterion [8] to illustrate correspondence between the EDM and positive semidefinite cones:

$$D \in \mathbb{EDM} \Leftrightarrow \begin{cases} -VDV \in \mathbb{S}_+ \\ D \in \mathbb{S}_h \end{cases} \tag{6}$$

where V is the geometric centering matrix

$$V \stackrel{\Delta}{=} I - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}} \in \mathbb{S}^N \tag{7}$$

in the ambient space of symmetric matrices $\mathbb{S}$ of dimension N.

2. Equality

Consider two convex cones $\mathscr{K}_1$ and $\mathscr{K}_2$ respectively defined

$$\begin{aligned} \mathscr{K}_1 &\stackrel{\Delta}{=} \mathbb{S}_h \\ \mathscr{K}_2 &\stackrel{\Delta}{=} \{A \in \mathbb{S} | -VAV \in \mathbb{S}_+\} \end{aligned} \tag{8}$$

so that

$$\mathscr{K}_1 \cap \mathscr{K}_2 = \mathbb{EDM} \tag{9}$$

Gaffke and Mathar [4, Section 5.3] observed that projection on $\mathscr{K}_1$ and $\mathscr{K}_2$ have simple closed forms: Projection on subspace $\mathscr{K}_1$ is easily performed by symmetrization and zeroing the main diagonal or *vice versa*, while projection of $H \in \mathbb{S}$ on $\mathscr{K}_2$ is

$$P_{\mathscr{K}_2} H = H - P_{\mathbb{S}_+}(VHV) \tag{10}$$

where $P_{\mathbb{S}_+}$ denotes projection on the positive semidefinite cone. Matrix product VHV is the orthogonal projection of H on the geometric center subspace $\mathbb{S}_c$. Thus the projection product

$$P_{\mathscr{K}_2} H = H - P_{\mathbb{S}_+} P_{\mathbb{S}_c} H \tag{11}$$

Because projection on the intersection of the positive semidefinite cone with the geometric center subspace is equivalent to a (noncommutative [3, Section 5.14]) projection product

$$P_{\mathbb{S}_+ \cap \mathbb{S}_c} = P_{\mathbb{S}_+} P_{\mathbb{S}_c} \tag{12}$$

a set equivalence follows:

$$\{P_{\mathbb{S}_+} P_{\mathbb{S}_c} H | H \in \mathbb{S}\} = \{P_{\mathbb{S}_+ \cap \mathbb{S}_c} H | H \in \mathbb{S}\} \tag{13}$$

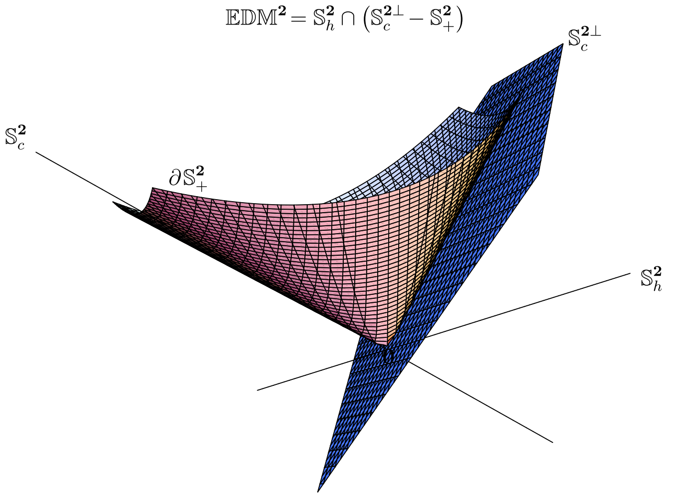

Fig. 1. EDM cone construction in isometrically isomorphic $\mathbb{R}^3$.

Polar cone $\mathcal{K}^\circ$ is a unique closed convex cone characterized by Moreau [7]: for closed convex cone $\mathcal{K}$

$$\begin{array}{c} x = x_1 + x_2, \quad x_1 \in \mathcal{K}, \quad x_2 \in \mathcal{K}^\circ, \quad x_1 \perp x_2 \\ \Leftrightarrow \\ x_1 = P_{\mathcal{K}} x, \quad x_2 = P_{\mathcal{K}^\circ} x \end{array} \tag{14}$$

which leads to concise cone relations

$$\begin{aligned} \mathcal{K} &\equiv \{P_{\mathcal{K}} x \mid x \in \mathbb{R}^N\} \\ \mathcal{K}^\circ &= \{x - P_{\mathcal{K}} x \mid x \in \mathbb{R}^N\} \end{aligned} \tag{15}$$

the former being obvious for any closed set $\mathcal{K}$. Thus

$$\begin{aligned} \mathbb{S}_c \cap \mathbb{S}_+ &\equiv \{P_{\mathbb{S}_+} P_{\mathbb{S}_c} H \mid H \in \mathbb{S}\} \\ (\mathbb{S}_c \cap \mathbb{S}_+)^\circ &= \{H - P_{\mathbb{S}_+} P_{\mathbb{S}_c} H \mid H \in \mathbb{S}\} \end{aligned} \tag{16}$$

Deutsch [3, Section 4.6] provides polar transformation of an intersection of closed convex cones to vector sum, from which

$$\mathcal{K}_2 = (\mathbb{S}_c \cap \mathbb{S}_+)^\circ = \mathbb{S}_c^\perp - \mathbb{S}_+ \tag{17}$$

because the subspace polar is its orthogonal complement, and the positive semidefinite cone is *self dual.* We therefore get the equality

$$\mathbb{EDM} = \mathcal{K}_1 \cap \mathcal{K}_2 = \mathbb{S}_h \cap (\mathbb{S}_c^\perp - \mathbb{S}_+) \tag{2}$$

whose veracity is intuitively evident, in hindsight [1, p. 109].

A realization of this construction in low dimension is illustrated in Fig. 1. Orthogonal complement $\mathbb{S}_c^{2\perp}$ (4) of the geometric center subspace (a plane in isometrically isomorphic $\mathbb{R}^3$; drawn is a tiled fragment) supports the positive semidefinite cone. (Rounded vertex is artifact of plot.) Line $\mathbb{S}_c^2$ runs along positive semidefinite cone boundary $\partial\mathbb{S}_+^2$. EDM cone construction is accomplished by adding the polar positive semidefinite cone to $\mathbb{S}_c^{2\perp}$. Difference $\mathbb{S}_c^{2\perp} - \mathbb{S}_+^2$ is a halfspace partially bounded by $\mathbb{S}_c^{2\perp}$. The EDM cone is a nonnegative halfline along $\mathbb{S}_h^2$ in this dimension.

3. Conclusion

Although its roots lie in the algebra of Schoenberg, we derived our main result (2) via established projection theory given by Moreau and by Deutsch. Equality (2) is a recipe for constructing the EDM cone whole from large Euclidean bodies: the positive semidefinite cone, orthogonal complement of the geometric center subspace, and symmetric hollow subspace.

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